Impulse and Linear Momentum

Linear Momentum

Linear momentum is a vector quantity that includes the effect of speed, mass and direction. (Often the term linear is omitted, but there is also a quantity angular momentum to be studied later.) The symbol is *p* and the units are kg•m/s. The equation for momentum is : $\vec{p} = m\vec{v}$. The direction of momentum and velocity are always the same. For a system of particles:

$\vec{P} = \vec{p}_1 + \vec{p}_2 + \vec{p}_3 + \cdots + \vec{p}_n = m\vec{v}_1 + m\vec{v}_2 + m\vec{v}_3 + \cdots + m\vec{v}_n$, therefore $\vec{P} = M\vec{v}_{cm}$. Taking the derivative:

$\frac{d\vec{P}}{dt} = M\frac{d\vec{v}}{dt} = M\vec{a}_{cm}$ thus proving $\sum \vec{F}_{ext} = \frac{d\vec{P}}{dt}$, which is the way Newton originally presented the second law of motion. For a single object: $\sum \vec{F}_{ext} = \frac{d\vec{p}}{dt} = m\frac{d\vec{v}}{dt} = m\vec{a}$

Impulse

Momentum can be changed by the application of a force. The effect of a force that changes momentum is called an impulse. When Newton stated the second law of motion, he said that the net external force on an object (or system) equals its rate of change in momentum.

$\sum \vec{F}_{ext} = \frac{d\vec{p}}{dt}$ If we have a function for the force in terms of time, we can write: *dp = F(t)dt*. Now if we separate the variables and integrate, we get: $\int_{p_i}^{p_f} d\vec{p} = \int_{t_i}^{t_f} \vec{F}(t)dt$. Evaluating the left side of the equation gives $p_f - p_i = \Delta p$. This is the impulse, the effect of the net external force over a period of time. The symbol is *J* and the units are the same as the units of momentum, kg•m/s. However, if impulse is calculated by using the force and time information, the units will be N•s, which is equivalent to kg•m/s. It is a vector, with the same direction as the net force and the change in momentum vector.

$\vec{J} = \int_{t_i}^{t_f} \vec{F}(t)dt$. If the force is constant or an average force is known, the equation can be simplified to: $J = F\Delta t$. Graphically, the impulse is the area between the curve and the time axis of a *F(t)* vs. *t* graph.

Conservation of Linear Momentum

If no external forces act on a closed system, momentum remains constant. Momentum can be transferred between objects within the system, but total momentum does not change. Since momentum is a vector, direction must be taken into account when momentum is calculated. Momentum may be conserved in one or two dimensions but not in all three e.g. projectiles. $\vec{p}_i = \vec{p}_f$

Collisions

A collision is an isolated event in which a relatively strong force acts on each of two or more colliding bodies for a relatively short time. This can be a gravitational, electrical or magnetic interaction where no actual contact takes place. The change in momentum for an object during a collision equals the impulse. The impulse for one body in a collision equals the impulse for the other body, but in the opposite direction (third law of motion). Thus, the change in momentum is also equal in magnitude for both colliding objects.

Two models are often used when studying collisions, the totally inelastic collision and the elastic collision. In a typical problem, the initial conditions are known and the final velocity or velocities are unknown. In the totally inelastic collision, the objects stick together after colliding and move as a combined mass. Momentum is conserved, but kinetic energy is not. Energy is lost through friction, sound, deformation and other dissipative forces. In one dimension, the equation for a totally inelastic collision is:

$$m_1 v_1 + m_2 v_2 = (m_1 + m_2) v_f$$

In the elastic collision, momentum and kinetic energy are both conserved. The objects rebound cleanly with no dissipative forces. Most actual collisions are somewhere in between these two models and are partially inelastic. In one dimension the equations for a totally elastic collision are:

$$m_1 v_{1i} + m_2 v_{2i} = m_1 v_{1f} + m_2 v_{2f} \quad \text{(cons. of momentum)}$$
$$\tfrac{1}{2} m_1 v_{1i}^2 + \tfrac{1}{2} m_2 v_{2i}^2 = \tfrac{1}{2} m_1 v_{1f}^2 + \tfrac{1}{2} m_2 v_{2f}^2 \quad \text{(cons. of kinetic energy)}$$

The kinetic energy equation can be simplified to become: $m_1 v_{1i}^2 + m_2 v_{2i}^2 = m_1 v_{1f}^2 + m_2 v_{2f}^2$

A third type of momentum transfer is a recoil or explosion event where an object composed of two parts springs apart due to a release of potential energy. The combined object is often at rest before the energy release. The two parts will recoil with equal and opposite momentum.

Special Cases of One-Dimensional Elastic Collisions

It is useful to examine several cases of elastic collisions. In one dimension with a stationary target, the equations become:

$$m_1 v_{1i} = m_1 v_{1f} + m_2 v_{2f}$$
$$m_1 v_{1i}^2 = m_1 v_{1f}^2 + m_2 v_{2f}^2$$

Solving for the final velocities we find that: $v_{1f} = \frac{m_1 - m_2}{m_1 + m_2} v_{1i}$ and $v_{2f} = \frac{2m_1}{m_1 + m_2} v_{1i}$ From this we can see that v_{2f} will always be positive, but v_{1f} could be positive or negative depending on the masses.

If the masses are equal, $v_{1f} = 0$ and $v_{2f} = v_{1i}$.

If $m_2 >> m_1$: $v_{1f} \approx -v_{1i}$ and $v_{2f} \approx \frac{2m_1}{m_2} v_{1i}$

If $m_1 >> m_2$: $v_{1f} \approx v_{1i}$ and $v_{2f} \approx 2v_{1i}$

The center of mass of the system continues moving unaffected by the collision:

$$v_{cm} = \frac{p_1 + p_2}{m_1 + m_2} = \frac{m_1 v_1}{m_1 + m_2} \text{ (target at rest)}$$

If the target is also moving, the situation becomes more complicated and the two conservation equations must be solved simultaneously. Often it is easier to examine these collisions from the reference frame of the center of mass. An interesting fact about elastic collisions is that they are symmetric with respect to the center of mass. If you stand at the center of mass to observe an elastic collision, you see mass m_1 approach with velocity V_1 (not the earth-frame-of-reference velocity v_1 above), and mass m_2 approaching with velocity V_2. The masses collide at the center of mass. Then, you see mass m_1 leaving the center of mass with velocity $-V_1$, and mass m_2 leaving the center of mass with velocity $-V_2$.

To find the velocities of the particles after the collision, you can:

1. Find the velocity of the system center of mass:

$$v_{cm} = \frac{m_1 v_1 + m_2 v_2}{m_1 + m_2}$$

2. Switch to the center of mass reference frame. To do this, simply subtract v_{cm} from each particle's velocity.
3. Have the collision. The particles' velocities reverse.
4. Switch back to the original frame of reference, by adding v_{cm} to each particle's velocity.

Actually, the solution for a one-dimensional elastic collision is even easier than that! It can be shown (quite easily) that

$v_{1f} = 2v_{cm} - v_{1i}$ and $v_{2f} = 2v_{cm} - v_{2i}$

Two dimensional collisions

In two (or three) dimensions, the vector nature of momentum makes the calculations more complicated. Each momentum vector must be resolved into *x*, *y*, (and *z*) components and solved separately. In the special case of equal masses and the target at rest the masses cancel out of both the momentum vector equation and the scalar kinetic energy equation. The result is:

$$\vec{v}_{1i} = \vec{v}_{1f} + \vec{v}_{2f}$$
$$v_{1i}^2 = v_{1f}^2 + v_{2f}^2$$

The vector equation solved graphically would produce a triangle composed of the three vectors. The scalar equation is actually the Pythagorean Theorem. This means that the angle between the two final velocity (and momentum) vectors is a right angle.