The following are equivalent statements about a real number $b$ and a polynomial $P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$.

- $x - b$ is a linear factor of the polynomial $P(x)$.
- $b$ is a zero of the polynomial function $y = P(x)$.
- $b$ is a root (or solution) of the polynomial equation $P(x) = 0$.
- $b$ is an $x$-intercept of the graph of $y = P(x)$.

**Theorem**  
**Factor Theorem**

The expression $x - a$ is a factor of a polynomial if and only if the value $a$ is a zero of the related polynomial function.

**Key Concept**  
**How Multiple Zeros Affect a Graph**

If $a$ is a zero of multiplicity $n$ in the polynomial function $y = P(x)$, then the behavior of the graph at the $x$-intercept $a$ will be close to linear if $n = 1$, close to quadratic if $n = 2$, close to cubic if $n = 3$, and so on.
**Problem 1**

**Writing a Polynomial Function in Factored Form**

What is the factored form of \( f(x) = x^3 - 2x^2 - 15x \)?

\[
x^3 - 2x^2 - 15x = x^3 - 2x^2 - 15x = x(x^2 - 2x - 15)
\]

Factor out the GCF, \( x \).

\[
= x(x - 5)(x + 3)
\]

Factor \( x^2 - 2x - 15 \).

\[
= x^3 - 2x^2 - 15x
\]

Distributive Property

**Check**  
\[ x(x - 5)(x + 3) = x^2 - 2x - 15 \]

Multiply \( (x - 5)(x + 3) \).

\[
= x^3 - 2x^2 - 15x
\]


**Problem 2**

**Finding Zeros of a Polynomial Function**

What are the zeros of \( f(x) = (x + 2)(x - 1)(x - 3) \)? Graph the function.

\[ f(x) = (x + 2)(x - 1)(x - 3) \]

\[ f(x) = (x + 2)(x - 1)(x - 3) \]

\[ f(x) = (x + 2)(x - 1)(x - 3) \]

\[ f(x) = (x + 2)(x - 1)(x - 3) \]

\[ f(x) = (x + 2)(x - 1)(x - 3) \]

**Know**
- Polynomial function

**Need**
- Zeros
- Additional points
- End behavior

**Plan**
- Use the Zero-Product Property to find zeros.
- Find points between the zeros.
- Sketch the graph.

**Step 1** Use the Zero-Product Property to find the zeros.

\[
(x + 2)(x - 1)(x - 3) = 0
\]

so \( x + 2 = 0 \) or \( x - 1 = 0 \) or \( x - 3 = 0 \).

The zeros of the function are \(-2, 1, 3\).

**Step 2** Find points for \( x \)-values between the zeros.

Evaluate \( f(x) = (x + 2)(x - 1)(x - 3) \) for \( x = -1, 0, \) and 2.

\[
(-1 + 2)(-1 - 1)(-1 - 3) = 8
\]

\[
(-1 - 2)(0 - 1)(0 - 3) = 6
\]

\[
(2 + 2)(2 - 1)(2 - 3) = -4
\]

**Step 3** Determine the end behavior.

The function \( f(x) = (x + 2)(x - 1)(x - 3) \) is cubic. The coefficient of \( x^3 \) is +1, so the end behavior is down and up.

**Step 4** Use the zeros: \(-2, 0, 1, 3\); the additional points: \((-1, 0), (0, 6), (2, -4)\); and end behavior to sketch the graph.
Writing a Polynomial Function From Its Zeros

A. What is a cubic polynomial function in standard form with zeros \(-2, 2,\) and \(3\)?

\[
f(x) = (x + 2)(x - 2)(x - 3)
\]

Write a linear factor for each zero.

\[
= (x + 2)(x^2 - 5x + 6)
\]

Multiply \((x - 2)\) and \((x - 3)\).

\[
= x(x^2 - 5x + 6) + 2(x^2 - 5x + 6)
\]

Distributive Property

\[
= x^3 - 5x^2 + 6x + 2x^2 - 10x + 12
\]

Distributive Property

\[
= x^3 - 3x^2 - 4x + 12
\]

Simplify.

The cubic polynomial \(f(x) = x^3 - 3x^2 - 4x + 12\) has zeros \(-2, 2,\) and \(3\).

B. What is a quartic polynomial function in standard form with zeros \(-2, -2, 2,\) and \(3\)?

\[
g(x) = (x + 2)(x + 2)(x - 2)(x - 3)
\]

Write a linear factor for each zero.

\[
= x^4 - x^3 - 10x^2 + 4x + 24
\]

Simplify.

The quartic polynomial \(g(x) = x^4 - x^3 - 10x^2 + 4x + 24\) has zeros \(-2, -2, 2,\) and \(3\).

C. Graph both functions. How do the graphs differ? How are they similar?

Both graphs have \(x\)-intercepts at \(-2, 2,\) and \(3\). The cubic has down-and-up end behavior with two turning points, and crosses the \(x\)-axis at \(-2\). The quartic has up-and-up end behavior, three turning points, and touches the \(x\)-axis at \(-2\) but does not cross it.
Problem 4

Finding the Multiplicity of a Zero

What is the factored form of \( f(x) = x^4 - 2x^3 - 8x^2 \)? What are the zeros? What are the multiplicities of the zeros? How does the graph behave at these zeros?

\[
f(x) = x^4 - 2x^3 - 8x^2 \\
= x^2(x^2 - 2x - 8) \quad \text{Factor out the GCF, } x^2. \\
= x^2(x + 2)(x - 4) \quad \text{Factor } (x^2 - 2x - 8).
\]

Since \( x^2 = (x - 0)^2 \), the number 0 is a zero of multiplicity 2.
The numbers \(-2\) and 4 are zeros of multiplicity 1.
The graph looks close to linear at the \(x\)-intercepts \(-2\) and 4.
It resembles a parabola at the \(x\)-intercept 0.

Problem 5

Identifying a Relative Maximum and Minimum

What are the relative maximum and minimum of \( f(x) = x^3 + 3x^2 - 24x \)?

Use a graphing calculator to find a relative maximum and a relative minimum.

The relative maximum is 80 at \( x = -4 \) and the relative minimum is \(-28\) at \( x = 2 \).
Using a Polynomial Function to Maximize Volume

**Technology** Digital box cameras connect to closed-circuit television and are used for security. Their design maximizes the volume while keeping the sum of the dimensions at 6 inches. If the length must be 1.5 times the height, what should each dimension be?

**Step 1** Define a variable \( x \).

Let \( x \) = the height of the camera.

**Step 2** Determine length and width.

length = 1.5\( x \); width = 6 − (\( x + 1.5x \)) = 6 − 2.5\( x \)

**Step 3** Model the volume.

\[
V = (\text{length})(\text{width})(\text{height}) = (1.5x)(6 - 2.5x)(x) = -3.75x^3 + 9x^2
\]

**Step 4** Graph the polynomial function. Use the **MAXIMUM** feature to find that the maximum volume is 7.68 in.\(^3\) for a height of 1.6 in.

height = \( x \) = 1.6

length = 1.5\( x \) = 1.5(1.6) = 2.4

width = 6 − 2.5\( x \) = 6 − 2.5(1.6) = 2

The dimensions of the camera should be 2.4 in. long by 2 in. wide by 1.6 in. high.

**PRACTICE and APPLICATION EXERCISES**

Write each polynomial function in factored form. Check by multiplication.

1. \( y = x^3 + 7x^2 + 10x \)
2. \( y = x^3 - 7x^2 - 18x \)
3. \( y = x^3 - 4x^2 - 21x \)
4. \( y = 3x^3 - 27x^2 + 24x \)
5. \( y = -2x^3 - 2x^2 + 40x \)
6. \( y = x^4 + 3x^3 - 4x^2 \)

Find the zeros of each function. Then graph the function.

7. \( y = (x - 1)(x + 2) \)
8. \( y = (x - 2)(x + 9) \)
9. \( y = x(x + 5)(x - 8) \)
10. \( y = (x + 1)(x - 2)(x - 3) \)
11. \( y = (x + 1)(x - 1)(x - 2) \)
12. \( y = x(x + 2)(x + 3) \)
13. **Apply Mathematics (1)(A)** A carpenter hollowed out the interior of a block of wood as shown at the right.

a. Express the volume of the original block and the volume of the wood removed as polynomials in factored form.

b. What polynomial represents the volume of the wood remaining?

Find the relative maximum and relative minimum of the graph of each function.

14. \( f(x) = x^3 + 4x^2 - 5x \)

15. \( f(x) = -x^3 + 16x^2 - 76x + 96 \)

16. \( f(x) = -4x^3 + 12x^2 + 4x - 12 \)

17. \( f(x) = x^3 - 7x^2 + 7x + 15 \)

Find the zeros of each function. State the multiplicity of multiple zeros.

18. \( y = (x + 3)^3 \)

19. \( y = x(x - 1)^3 \)

20. \( y = 2x^3 + x^2 - x \)

21. \( y = 3x^3 - 3x \)

22. \( y = (x - 4)^2 \)

23. \( y = (x - 2)^2(x - 1) \)

24. \( y = (2x + 3)(x - 1)^2 \)

25. \( y = (x + 1)^2(x - 1)(x - 2) \)

26. **Use a Problem-Solving Model (1)(B)** A storage company needs to design a new storage box that has twice the volume of its largest box. Its largest box is 5 ft long, 4 ft wide, and 3 ft high. The new box must be formed by increasing each dimension by the same amount. Find the increase in each dimension.

27. Write a polynomial function in standard form with the given zeros.

28. \( x = -2, 0, 1 \)

29. \( x = -5, -5, 1 \)

30. \( x = 3, 3, 3 \)

31. \( x = 1, -1, -2 \)

32. \( x = 0, 4, \frac{-1}{2} \)

33. \( x = 0, 0, 2, 3 \)

34. \( x = -1, -2, -3, -4 \)

35. Write a polynomial function with the following features: it has three distinct zeros; one of the zeros is 1; another zero has a multiplicity of 2.

36. **Explain Mathematical Ideas (1)(G)** Explain how the graph of a polynomial function can help you factor the polynomial.

37. **Apply Mathematics (1)(A)** A metalworker wants to make an open box from a sheet of metal, by cutting equal squares from each corner as shown.

a. Write expressions for the length, width, and height of the open box.

b. Use your expressions from part (a) to write a function for the volume of the box. (Hint: Write the function in factored form.)

c. Graph the function. Then find the maximum volume of the box and the side length of the cut-out squares that generates this volume.
38. **Apply Mathematics (1)(A)** A rectangular box is $2x + 3$ units long, $2x - 3$ units wide, and $3x$ units high. What is its volume, expressed as a polynomial?

39. **Apply Mathematics (1)(A)** The volume in cubic feet of a CD holder can be expressed as $V(x) = -x^3 - x^2 + 6x$, or, when factored, as the product of its three dimensions. The depth is expressed as $2 - x$. Assume that the height is greater than the width.
   a. Factor the polynomial to find linear expressions for the height and the width.
   b. Graph the function. Find the $x$-intercepts. What do they represent?
   c. What is a realistic domain for the function?
   d. What is the maximum volume of the CD holder?

40. Find a fourth-degree polynomial function with zeros $1, -1, i,$ and $-i$. Write the function in factored form.

41. a. Compare the graphs of $y = (x + 1)(x + 2)(x + 3)$ and $y = (x - 1)(x - 2)(x - 3)$. What transformation could you use to describe the change from one graph to the other?
   b. Compare the graphs of $y = (x + 1)(x + 3)(x + 7)$ and $y = (x - 1)(x - 3)(x - 7)$. Does the transformation that you chose in part (a) still hold true? Explain.
   c. What transformation could you use to describe the effect of changing the signs of the zeros of a polynomial function?

### TEXAS Test Practice

42. The three most frequent letters in the English language are E, T, and A. They represent, on average, 30% of all letters. The most frequent letter, E, is 4% more frequent than the second most frequent letter, T. The combined frequency of T and A is 4% greater than the frequency of E. Approximately how many E’s can you expect to encounter in a 500-letter paragraph?
   A. 49  B. 65  C. 72  D. 88

43. Which expression is the factored form of $x^3 + 2x^2 - 5x - 6$?
   F. $(x + 1)(x + 1)(x - 6)$  H. $(x + 2)(2x - 5)(x - 6)$
   G. $(x + 3)(x + 1)(x - 2)$  J. $(x - 3)(x - 1)(x + 2)$

44. A ball with a 3 in. radius has volume $V_1$. A second ball has a 9 in. radius and volume $V_2$. Which equation represents the volume of the second ball in terms of the first?
   A. $V_2 = 3V_1$  B. $V_2 = 27V_1$  C. $V_2 = V_1^2$  D. $V_2 = 9V_1^2$

45. What is the polynomial function, in factored form, whose zeros are $-2, 5,$ and $6$, and whose leading coefficient is $-2$? Graph this function and find any relative minimums or maximums.